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VARIANCE OF RANDOM SIGNAL MEAN SQUARE VALUE DIGITAL ESTIMATOR

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Abstract

In the article, original relations enabling the estimation of the variance of a random signal mean square value digital estimator are derived. Three cases are considered: first when the estimator is determined from quantized samples; second, when it is additionally assumed that the conditions of Widrow's theorem are satisfied; and third, when the samples have not been quantized. The obtained relations can be used *e.g.* to determine uncertainty in precision measurements and to evaluate signal degradation in radio astronomy. As an example, the variance of the mean square value estimator of a random Gaussian signal for the three above-mentioned situations is analyzed. It has been observed that in the first and second cases, an increase in variance as well as in type A standard uncertainty takes place in comparison with the estimation based on unquantized samples. This increase diminishes along with an increase in the ratio of the signal rms value to the quantization step size.

Keywords: mean square value, digital estimator, characteristic function, estimator variance, type A uncertainty.

1. Introduction

The issue of signal quantization and quantization errors is the subject of numerous publications, *e.g.* [1]-[7]. Bernard Widrow is considered as the creator of the signal quantization theory. In articles from the end of the 1950s and the beginning of the 1960s, Widrow formulated several important theorems concerning the reconstruction conditions of the moments of a random variable corresponding to the quantized data [8, 9]. The great interest during the 1980s and 1990s in A/D conversion and the problem of reconstruction conditions for quantization voiced in prestigious journals, induced Widrow to formulate the theorems in terms of modern language [10], and metrologists to express their attitude toward them [11]-[18].

The mean square value is next to the mean value an important signal parameter. Determining it from quantized data is the source of estimation errors: bias and variance. Bias has been analyzed in detail and described in [6], [7], [19]-[20]. Variance determines within what range values concentrated in a series fluctuate around the series mean and provides a quantitative measure of these fluctuations [22]. In metrology, the square root serves as a means of evaluating the scattering of the results around the mean as well as type A uncertainty. Using variance, one can evaluate the quantization effects and the usefulness of signal averaging algorithms.

Presented in the article are results of the analyses of the mean square value digital estimator of a random signal quantized in a roundoff quantizer.

2. Object of study

A discrete random signal is sampled in time and quantized in amplitude. Let uncorrelated and grouped samples of the signal constitute an *M*-dimensional random variable $x_q = \left(x_q(0), x_q(1), ..., x_q(M-1)\right)$, where *M* is the number of samples [23].

The mean square of x_q is a typical statistic. In this paper, the mean square of x_q is marked by $\tilde{\tilde{x}}_q^2$ and calculated from the formula

$$\tilde{\tilde{x}}_q^2 = \frac{1}{M} \sum_{n=0}^{M-1} \left(x_q(n)\right)^2. \tag{1}$$

In metrology nomenclature, the statistic $\tilde{\tilde{x}}_q^2$, which is an estimator of the true value $\bar{x}^2$ of the signal mean square, is called a mean square value digital estimator.

Also studied in the article is a statistic $\tilde{\tilde{x}}^2$, which is a function of a random variable $x = \left(x(0), x(1), ..., x(M-1)\right)$ created from discrete random signal samples. The statistic $\tilde{\tilde{x}}^2$ is an estimator of the true value $\bar{x}^2$ of the signal mean square.

3. Determination of mean square value digital estimator variance

The implementation in digital devices of finite word length A/D converters is a source of inevitable errors called quantization errors.

Let an A/D converter be an ideal converter; let the quantization error probability density function (PDF) $e_q = x_q - x$ be monotonous; and let all possible errors between and $q/2$ be equiprobable, where q is the quantization step size [22].

The expectation value of the estimator $\tilde{\tilde{x}}_q^2$ is

$$\begin{aligned} E\left[\tilde{\tilde{x}}_q^2\right] &= \frac{1}{M} \sum_{n=0}^{M-1} E\left[\left(x_q(n)\right)^2\right] = \frac{1}{M}\left(ME\left[x_q^2\right]\right) = E\left[x_q^2\right] \\ &= E\left[\left(x+e_q\right)^2\right] = E\left[x^2\right] + 2E\left[xe_q\right] + E\left[e_q^2\right]. \end{aligned} \tag{2}$$

The variance of the estimator $\tilde{\tilde{x}}_q^2$ can be calculated from the definition

$$Var\left[\tilde{\tilde{x}}_q^2\right] = E\left[\left(\tilde{\tilde{x}}_q^2 - E\left[\tilde{\tilde{x}}_q^2\right]\right)^2\right] = E\left[\left(\tilde{\tilde{x}}_q^2\right)^2\right] - \left(E\left[\tilde{\tilde{x}}_q^2\right]\right)^2. \tag{3}$$

It can be noted that

$$\begin{aligned} E\left[\left(\tilde{\tilde{x}}_q^2\right)^2\right] &= \frac{1}{M^2} E\left[\left(\sum_{n=0}^{M-1}\left(x_q(n)\right)^2\right)^2\right] \\ &= \frac{1}{M^2}\left(\sum_{n=0}^{M-1} E\left[\left(x_q(n)\right)^4\right] + \sum_{n=0}^{M-1}\sum_{\substack{k=0 \\ n\neq k}}^{M-1} E\left[\left(x_q(n)\right)^2\left(x_q(k)\right)^2\right]\right) \\ &= \frac{1}{M^2}\left(ME\left[x_q^4\right] + M(M-1)\left(E\left[x_q^2\right]\right)^2\right) \\ &= \frac{1}{M}\left(E\left[x_q^4\right] - \left(E\left[x_q^2\right]\right)^2\right) + \left(E\left[x_q^2\right]\right)^2. \end{aligned} \tag{4}$$

Substituting the relations (2) and (4) in the formula (3) we obtain

$$Var\left[\tilde{\tilde{x}}_q^2\right]=\frac{1}{M}\left(E\left[x_q^4\right]-\left(E\left[x_q^2\right]\right)^2\right), \tag{5}$$

where

$$E\left[x_q^4\right]=E\left[x^4\right]+4E\left[x^3e_q\right]+6E\left[x^2e_q^2\right]+4E\left[xe_q^3\right]+E\left[e_q^4\right]. \tag{6}$$

The formulas (2) and (6) are relations linking the 2nd and 4th moments of x_q with the 2nd and 4th moments of x.

Finally, the variance of the estimator $\tilde{\tilde{x}}_q^2$ can be calculated from

$$Var\left[\tilde{\tilde{x}}_q^2\right]=\frac{1}{M}\left(E\left[x^4\right]+4E\left[x^3e_q\right]+6E\left[x^2e_q^2\right]+4E\left[xe_q^3\right]+E\left[e_q^4\right]-\left(E\left[x^2\right]+2E\left[xe_q\right]+E\left[e_q^2\right]\right)^2\right). \tag{7}$$

It can easily be noted that for $M \to \infty$, the variance $Var\left[\tilde{\tilde{x}}_q^2\right] \to 0$. Moreover, for $q \to 0$, $Var\left[\tilde{\tilde{x}}_q^2\right]\to Var\left[\tilde{\tilde{x}}^2\right]$ takes place, which agrees with intuition.

It can be shown that the estimator $\tilde{\tilde{x}}^2$ is not biased, *i.e.* $E\left[\tilde{\tilde{x}}^2\right] = \bar{x}^2$, and that the mean square error of this estimator is equal to the variance

$$Var\left[\tilde{\tilde{x}}^2\right]=\frac{1}{M}\left(E\left[x^4\right]-\left(E\left[x^2\right]\right)^2\right). \tag{8}$$

This means that for signals undergoing quantization in an infinite-resolution converter, the variance of the estimator $\tilde{\tilde{x}}_q^2$ does not depend on the moments of x_q, but on the moments of x. The formula (8) is presented in the literature as a particular case of the variance of a quadratic form [24].

The *m*-th moments of x_q can be calculated by differentiating its characteristic function [10]

$$E\left[x_q^m\right]=(-j)^m\left.\frac{d^m\Phi_{x_q}(v)}{dv^m}\right|_{v=0}, \; j=\sqrt{-1}, \tag{9}$$

where

$$\Phi_{x_q}(v)=\sum_{i=-\infty}^{\infty}\Phi_x\left(v-\frac{2\pi}{q}i\right)\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right), \tag{10}$$

and $\Phi_x(v)=E\left[e^{jvx}\right]$ is the characteristic function, the Fourier transform of the PDF of x [24] and

$$\operatorname{sinc}(\alpha)=\begin{cases}\dfrac{\sin(\alpha)}{\alpha}, & \alpha\neq 0,\\ 1, & \alpha=0.\end{cases} \tag{11}$$

Using (9), the following simpler expression can be obtained

$$E\left[x_q^2\right] = -\left.\frac{d^2\Phi_{x_q}(v)}{dv^2}\right|_{v=0} . \tag{12}$$

Combining the formulas (10) and (12), we obtain

$$\begin{aligned} E\left[x_q^2\right] &= -\left.\frac{d^2\Phi_{x_q}(v)}{dv^2}\right|_{v=0} \\ &= -\sum_{i=-\infty}^{\infty} \frac{d^2}{dv^2}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0} \operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} \\ &\quad -2\sum_{i=-\infty}^{\infty} \frac{d}{dv}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0} \frac{d}{dv}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} \\ &\quad -\sum_{i=-\infty}^{\infty} \Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0} \frac{d^2}{dv^2}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} . \end{aligned} \tag{13}$$

Because

$$\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)_{(v,i)=(0,0)} = 1, \tag{14}$$

$$\frac{d}{dv}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{(v,i)=(0,0)} = 0, \tag{15}$$

$$\frac{d^2}{dv^2}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{(v,i)=(0,0)} = -\frac{q^2}{12}, \tag{16}$$

we obtain

$$\begin{aligned} E\left[x_q^2\right] = &-\frac{d^2}{dv^2}\Phi_x(v)\bigg|_{v=0} - 4\sum_{i=1}^{\infty}\frac{d}{dv}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0}\frac{d}{dv}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} \\ &-2\sum_{i=1}^{\infty}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0}\frac{d^2}{dv^2}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} + \frac{q^2}{12}. \end{aligned} \tag{17}$$

Since $\sin(\pi i) = 0$ and $\cos(\pi i) = (-1)^i$, $i = 1, 2, \ldots$, then the derivatives of the function sinc assume the form

$$\frac{d}{dv}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} = -\frac{1}{2}\left(\frac{q}{\pi}\right)\frac{(-1)^i}{i}, \tag{18}$$

$$\frac{d^2}{dv^2}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} = -\frac{1}{2}\left(\frac{q}{\pi}\right)^2\frac{(-1)^i}{i^2}. \tag{19}$$

Substituting (18) and (19) in (17), we obtain

$$E\left[x_q^2\right] = E\left[x^2\right] + \frac{2q}{\pi}\sum_{i=1}^{\infty}\frac{d}{dv}\Phi_x(v)\Big|_{v=\frac{2\pi}{q}i}\frac{(-1)^{i+1}}{i} + \frac{q^2}{\pi^2}\sum_{i=1}^{\infty}\Phi_x\left(\frac{2\pi}{q}i\right)\frac{(-1)^i}{i^2} + \frac{q^2}{12}. \tag{20}$$

Equating sides of the formulas (2) and (20), we obtain [3, 25, 26]

$$E\left[xe_q\right] = \frac{q}{\pi}\sum_{i=1}^{\infty}\frac{d}{dv}\Phi_x(v)\Big|_{v=\frac{2\pi}{q}i}\frac{(-1)^{i+1}}{i}, \tag{21}$$

$$E\left[e_q^2\right] = \frac{q^2}{\pi^2}\sum_{i=1}^{\infty}\Phi_x\left(\frac{2\pi}{q}i\right)\frac{(-1)^i}{i^2} + \frac{q^2}{12}. \tag{22}$$

The higher moments can be determined in a similar way. The 4th moment is as follows

$$\begin{aligned} E\left[x_q^4\right] &= \frac{d^4\Phi_{x_q}(v)}{dv^2}\Bigg|_{v=0} \\ &= \sum_{i=-\infty}^{\infty}\frac{d^4}{dv^4}\Phi_x\left(v-\frac{2\pi}{q}i\right)\Bigg|_{v=0}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{v=0} \\ &+ 4\sum_{i=-\infty}^{\infty}\frac{d}{dv}\Phi_x\left(v-\frac{2\pi}{q}i\right)\Bigg|_{v=0}\frac{d^3}{dv^3}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{v=0} \\ &+ 6\sum_{i=-\infty}^{\infty}\frac{d^2}{dv^2}\Phi_x\left(v-\frac{2\pi}{q}i\right)\Bigg|_{v=0}\frac{d^2}{dv^2}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{v=0} \\ &+ 4\sum_{i=-\infty}^{\infty}\frac{d^3}{dv^3}\Phi_x\left(v-\frac{2\pi}{q}i\right)\Bigg|_{v=0}\frac{d}{dv}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{v=0} \\ &+ \sum_{i=-\infty}^{\infty}\Phi_x\left(v-\frac{2\pi}{q}i\right)\Bigg|_{v=0}\frac{d^4}{dv^4}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{v=0}. \end{aligned} \tag{23}$$

Making use of the formulas (14)-(16) as well as of the fact that

$$\frac{d^3}{dv^3}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{(v,i)=(0,0)} = 0, \tag{24}$$

$$\frac{d^4}{dv^4}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\Bigg|_{(v,i)=(0,0)} = \frac{q^4}{80}, \tag{25}$$

after ordering, we obtain

$$
\begin{aligned}
E\left[x_q^4\right] &= \frac{d^4}{dv^4}\Phi_x(v)\bigg|_{v=0} + 8\sum_{i=1}^{\infty}\frac{d}{dv}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0}\frac{d^3}{dv^3}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} \\
&+12\sum_{i=1}^{\infty}\frac{d^2}{dv^2}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0}\frac{d^2}{dv^2}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} - \frac{q^2}{2}\frac{d^2}{dv^2}\Phi_x(v)\bigg|_{v=0} \\
&+8\sum_{i=1}^{\infty}\frac{d^3}{dv^3}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0}\frac{d}{dv}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} \\
&+2\sum_{i=1}^{\infty}\Phi_x\left(v-\frac{2\pi}{q}i\right)\bigg|_{v=0}\frac{d^4}{dv^4}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} + \frac{q^4}{80}.
\end{aligned} \tag{26}
$$

Making use of the formulas (18)-(19) as well as of the fact that

$$
\frac{d^3}{dv^3}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} = \frac{q^2}{8}\left(\frac{q}{\pi}\right)\frac{(-1)^i}{i} - \frac{3}{4}\left(\frac{q}{\pi}\right)^3\frac{(-1)^i}{i^3}, \tag{27}
$$

$$
\frac{d^4}{dv^4}\operatorname{sinc}\left(\frac{qv}{2}-\pi i\right)\bigg|_{v=0} = \frac{q^2}{4}\left(\frac{q}{\pi}\right)^2\frac{(-1)^i}{i^2} - \frac{3}{2}\left(\frac{q}{\pi}\right)^4\frac{(-1)^i}{i^4}, \tag{28}
$$

we obtain

$$
\begin{aligned}
E\left[x_q^4\right] &= E\left[x^4\right] + \frac{4q}{\pi}\sum_{i=1}^{\infty}\frac{d^3}{dv^3}\Phi_x(v)\bigg|_{v=\frac{2\pi}{q}i}\frac{(-1)^i}{i} + \frac{6q^2}{\pi^2}\sum_{i=1}^{\infty}\frac{d^2}{dv^2}\Phi_x(v)\bigg|_{v=\frac{2\pi}{q}i}\frac{(-1)^{i+1}}{i^2} + \frac{q^2}{2}E\left[x^2\right] \\
&+\frac{2q^3}{\pi}\sum_{i=1}^{\infty}\frac{d}{dv}\Phi_x(v)\bigg|_{v=\frac{2\pi}{q}i}\left(\frac{1}{2}-\frac{3}{\pi^2 i^2}\right)\frac{(-1)^{i+1}}{i} + \frac{q^4}{\pi^2}\sum_{i=1}^{\infty}\Phi_x\left(\frac{2\pi}{q}i\right)\left(\frac{1}{2}-\frac{3}{\pi^2 i^2}\right)\frac{(-1)^i}{i^2} + \frac{q^4}{80}.
\end{aligned} \tag{29}
$$

Equating sides of the formulas (6) and (29), we obtain:

$$
E\left[x^3 e_q\right] = \frac{q}{\pi}\sum_{i=1}^{\infty}\frac{d^3}{dv^3}\Phi_x(v)\bigg|_{v=\frac{2\pi}{q}i}\frac{(-1)^i}{i}, \tag{30}
$$

$$
E\left[x^2 e_q^2\right] = \frac{q^2}{\pi^2}\sum_{i=1}^{\infty}\frac{d^2}{dv^2}\Phi_x(v)\bigg|_{v=\frac{2\pi}{q}i}\frac{(-1)^{i+1}}{i^2} + \frac{q^2}{12}E\left[x^2\right], \tag{31}
$$

$$
E\left[x e_q^3\right] = \frac{q^3}{2\pi^3}\sum_{i=1}^{\infty}\frac{d}{dv}\Phi_x(v)\bigg|_{v=\frac{2\pi}{q}i}\left(\frac{1}{2}\pi^2 i^2 - 3\right)\frac{(-1)^{i+1}}{i^3}, \tag{32}
$$

$$
E\left[e_q^4\right] = \frac{q^4}{\pi^4}\sum_{i=1}^{\infty}\Phi_x\left(\frac{2\pi}{q}i\right)\left(\frac{1}{2}\pi^2 i^2 - 3\right)\frac{(-1)^i}{i^4} + \frac{q^4}{80}. \tag{33}
$$

The relations (21), (22) and (30)-(33) make it possible to determine the variance (7).

Let us now refer to Widrow's theorem [10]:

Theorem (Widrow's). If a characteristic function has a limited domain, *i.e.*

$$\Phi_x(v) = 0, \text{ for } |v| > \frac{2\pi}{q} - \varepsilon, \tag{34}$$

where ε is an infinitely small positive number, then the moments of x can be calculated from the moments of x_q.

When the premises of Widrow's theorem are satisfied, then the expressions (2) and (6) assume the form [10]

$$E[x_q^2] = E[x^2] + \frac{q^2}{12}, \tag{35}$$

$$E[x_q^4] = E[x^4] + \frac{q^2}{2} E[x^2] + \frac{q^4}{80}. \tag{36}$$

From the formula (5) and the formulas (35) and (36), we can compute

$$Var_{sh}\left[\tilde{\bar{x}}_q^2\right] = \frac{1}{M}\left(E[x^4] - \left(E[x^2]\right)^2 + \frac{1}{3} E[x^2] q^2 + \frac{1}{180} q^4 \right), \tag{37}$$

where $Var_{sh}\left[\tilde{\bar{x}}_q^2\right]$ denotes the mean square value estimator variance when the premises of Widrow's theorem are satisfied, *i.e.* with the so-called Sheppard's corrections taken into account.

Shown below is an example of calculating the variance and the uncertainty of the mean square value digital estimator of a random Gaussian signal.

EXAMPLE - VARIANCE AND TYPE A UNCERTAINTY OF NORMAL DISTRIBUTION SIGNAL MEAN SQUARE VALUE DIGITAL ESTIMATOR

The moments corresponding to a Gaussian variable are: $E[x^2] = \sigma_x^2$ and $E[x^4] = 3\sigma_x^4$, where σ_x is the standard deviation of x [24].

From the relations (8) we obtain a formula for the variance of the estimator $\tilde{\bar{x}}^2$ of the parameter $\bar{x}^2$ [27]

$$Var\left[\tilde{\bar{x}}^2\right] = \frac{1}{M}\left(3\sigma_x^4 - \sigma_x^4\right) = \frac{2}{M}\sigma_x^4. \tag{38}$$

The variance of the estimator $\tilde{\bar{x}}_q^2$ can be written as in (7)

$$Var\left[\tilde{\bar{x}}_q^2\right] = \frac{1}{M}\left(3\sigma_x^4 + 4E[x^3 e_q] + 6E[x^2 e_q^2] + 4E[x e_q^3] + E[e_q^4] - \left(\sigma_x^2 + 2E[x e_q] + E[e_q^2]\right)^2\right) \tag{39}$$

The expressions $E[xe_q]$, $E[e_q^2]$, $E[x^3 e_q]$, $E[x^2 e_q^2]$, $E[x e_q^3]$ can be determined from the formulas (21), (22) and (30)-(33). The characteristic function of a random Gaussian variable present in the formulas assumes the form [24]

$$\Phi_x(v) = \exp\left(-0.5 v^2 \sigma_x^2\right). \tag{40}$$

When the conditions of Widrow's theorem[1] are met, the variance of the estimator $\tilde{\tilde{x}}_q^2$ can be computed from the equation (37).

For a random Gaussian variable, we obtain

$$Var_{sh}\left[\tilde{\tilde{x}}_q^2\right] = \frac{1}{M}\left(2\sigma_x^4 + \frac{1}{3}\sigma_x^2 q^2 + \frac{1}{180}q^4\right). \tag{41}$$

The relations (38), (39) and (41) can be normalized relative to the square of the mean square value $\left(\sigma_x^2\right)^2 = \sigma_x^4$, which will facilitate their comparison.

$$\xi^2 = \frac{Var\left[\tilde{\tilde{x}}^2\right]}{\sigma_x^4} = \frac{2}{M}, \tag{42}$$

$$\xi_q^2 = \frac{Var\left[\tilde{\tilde{x}}_q^2\right]}{\sigma_x^4}, \tag{43}$$

$$\xi_{qsh}^2 = \frac{Var_{sh}\left[\tilde{\tilde{x}}_q^2\right]}{\sigma_x^4} = \frac{2}{M}\left[1 + \frac{1}{6}\left(\frac{\sigma_x}{q}\right)^{-2} + \frac{1}{360}\left(\frac{\sigma_x}{q}\right)^{-4}\right]. \tag{44}$$

Shown in Table 1 are the example values ξ_q^2, ξ_{qsh}^2 for $M = 1\,000$. The value ξ^2 for the number of samples $M = 1\,000$ is equal to $2 \cdot 10^{-3}$.

Table 1. Normalized variances ξ_q^2, ξ_{qsh}^2 for $M = 1\,000$.

σ_n / q	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
ξ_q^2	0.0112	0.00404	0.00262	0.00234	0.00222	0.00215	0.00211	0.00209
ξ_{qsh}^2	0.00876	0.00343	0.00261	0.00234	0.00222	0.00215	0.00211	0.00209

The normalized variances are given by formulas (43)-(45) as a function of σ_x / q and shown in Fig. 1.

[1] Further in the article, we assess the consequences of such an assumption.

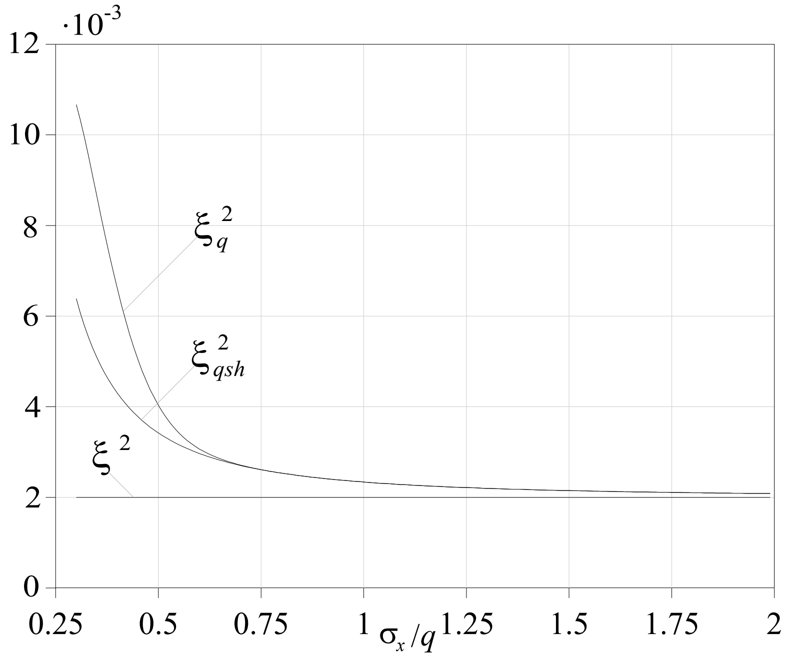

Fig. 1. Normalized variances: ξ^2, ξ_q^2, ξ_{qsh}^2.

As follows from Table 1 and Fig. 1, the normalized variances ξ_q^2 and ξ_{qsh}^2 approach the value $2/M$ along with an increase in σ_x / q. In practice, for $\sigma_x / q \geq 1.00$, their values are equal and exceed the value of ξ^2 respectively by 17% for $\sigma_x = 1.00$, and by 4.5% for $\sigma_x = 2.00$.

In measurements, type A uncertainty determines a scatter of measurement results and is proportional to the square root of the estimator variance (standard deviation). Knowing the variance can therefore make it possible to evaluate the effect of quantization on the level of this uncertainty.

Let us introduce the coefficients

$$k_q = \frac{\sqrt{Var\left[\tilde{\tilde{x}}_q^2\right]}}{\sqrt{Var\left[\tilde{\tilde{x}}^2\right]}}, \tag{45}$$

$$k_{qsh} = \frac{\sqrt{Var_{sh}\left[\tilde{\tilde{x}}_q^2\right]}}{\sqrt{Var\left[\tilde{\tilde{x}}^2\right]}} = \sqrt{1 + \frac{1}{6}\left(\frac{\sigma_x}{q}\right)^{-2} + \frac{1}{360}\left(\frac{\sigma_x}{q}\right)^{-4}}, \tag{46}$$

which can be interpreted as: k_q - the increase in type A standard uncertainty due to signal quantization, k_{qsh} - the increase in type A standard uncertainty due to signal quantization and to the assumption that the moments of a random variable corresponding to the quantized signal satisfy the premises of Widrow's theorem. Shown in Table 2 and in Fig. 2 are the coefficient values as a function of σ_x / q. They make it possible to estimate uncertainty as a multiple of the expression $\sqrt{Var\left[\tilde{\tilde{x}}^2\right]} = \sqrt{\frac{2}{M}}\sigma_x^2$.

Table 2. Values of the coefficients k_q , k_{qsh} .

σ_x / q	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
k_q	2.36	1.42	1.15	1.09	1.06	1.04	1.03	1.02
k_{qsh}	2.10	1.31	1.15	1.09	1.06	1.04	1.03	1.02

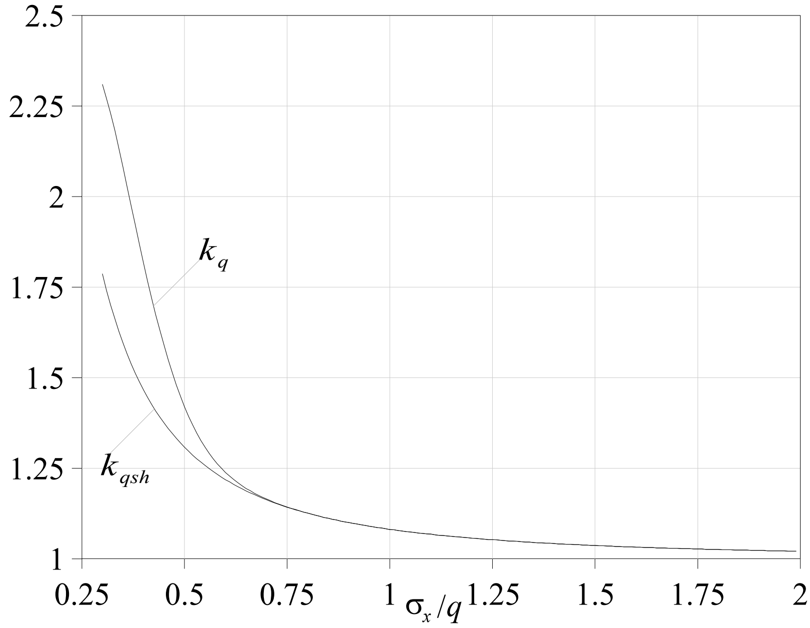

Fig. 2. Coefficients k_q , k_{qsh} as a function of σ_x / q .

From Table 2 and Fig. 2, it follows that the coefficients k_q and k_{qsh} for $\sigma_x / q \geq 0.75$ assume similar values, which means that the values of type A standard uncertainty calculated using the relations $\sqrt{Var\left[\tilde{\tilde{x}}_q^2\right]}$ and $\sqrt{Var_{sh}\left[\tilde{\tilde{x}}_q^2\right]}$ practically do not differ. Moreover, for $\sigma_x / q = 1.00$ and 2.00, the values of uncertainty differ respectively by 9% and 2% from those calculated without taking into account the quantization process of the signals, i.e. using the relation $\sqrt{Var\left[\tilde{\tilde{x}}^2\right]}$.

4. Conclusions

In the article, the original relations (7) and (37) enabling the estimation of the variance of a random signal mean square value digital estimator have been derived. Three cases have been considered: first, when the estimator is determined from quantized samples; second, when additionally it is assumed that the conditions of Widrow's theorem are satisfied; and third, when the samples have not been quantized. The obtained relations can be used to evaluate type A measurement uncertainty, which requires that the characteristic function corresponding to the analyzed signal be known.

As an example, the variance and type A uncertainty of the mean square value estimator of a random Gaussian signal in the three above mentioned situations have been analyzed. It has been observed that in the first and second cases, an increase in variance as well as in type A standard uncertainty takes place relative to the estimation from unquantized samples. This increase diminishes along with an increase in the ratio of the signal rms value to the quantization step size. The assumption that a signal satisfies the premises of the quantization theorem generally causes an underestimation of the variance and of type A uncertainty of the digital estimator, but for $\sigma_x / q \geq 1.00$ this underestimation may in practice be negligible.

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